Borel summation of the small time expansion of the heat kernel. The scalar potential case*†

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Abstract

Let p_t be the heat kernel associated to the operator $-\Delta+V(x)$ defined on $\mathbb{R}^\nu$. We prove, under rather strong assumptions on V, that the small time expansion of p_t is Borel summable. If V is defined on the torus, we prove a Poisson formula.

1 Introduction

Let $\nu \in \mathbb{N}^*$ and V be a regular square matrix-valued function on $\mathbb{R}^\nu$. Denote $\partial_x^2 := \partial_{x_1}^2 + \cdots + \partial_{x_\nu}^2$ and $(x-y)^2 := (x_1-y_1)^2 + \cdots + (x_\nu - y_\nu)^2$ for $x \in \mathbb{R}^\nu$and $y \in \mathbb{R}^\nu$. Let $p_t(x,y)$ be the heat kernel associated to the operator $-\partial_x^2 + V(x)$. Let $p_t^{\text{conj}}(x,y)$ be the conjugate heat kernel defined by

$$p_t(x,y) = (4\pi t)^{-\frac{\nu}{2}} \exp\Big(-\frac{(x-y)^2}{4t}\Big) p_t^{\text{conj}}(x,y). \tag{1.1}$$

Then the Minakshisundaram-Pleijel asymptotic expansion holds:

$$p_t^{\text{conj}}(x,y) = \mathbb{1} + a_1(x,y)t + \cdots + a_{r-1}(x,y)t^{r-1} + t^r \mathcal{O}_{t\to 0^+}(1). \tag{1.2}$$

Here $\mathbb{1}$ denotes the identity matrix. The expansion in (1.2) is not convergent in general. A goal of this paper is to prove, under rather strong assumptions on V, that this expansion is Borel summable and that its Borel sum is equal to $p_t^{\text{conj}}(x,y)$ (see definition 2.4). Borel summability allows to recover $p_t^{\text{conj}}(x,y)$ with the help of the knowledge of the coefficients $a_1(x,y), a_2(x,y), \ldots$ For instance, if these coefficients vanish then $p_t^{\text{conj}}(x,y) = \mathbb{1}$.

Assume now that V is defined on the torus $(\mathbb{R}/\mathbb{Z})^\nu$ with values in a space of $d \times d$ Hermitian matrices. Let $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots, \lambda_n \to +\infty$ be the

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eigenvalues of the operator$-\partial_x^2 + V(x)$ acting on periodic $\mathbb{C}^d$-valued functions. The trace of the heat kernel has an asymptotic expansion

$$\sum_{n=1}^{+\infty} e^{-\lambda_n t} = (4\pi t)^{-\frac{\nu}{2}} \big(d + a_1 t + \cdots + a_{r-1} t^{r-1} + t^r \mathcal{O}_{t\to 0^+}(1)\big). \tag{1.3}$$

We shall prove the Poisson formula: for $t \in \mathbb{C}$, $\mathcal{R}et > 0$,

$$\sum_{n=1}^{+\infty} e^{-\lambda_n t} = (4\pi t)^{-\frac{\nu}{2}} \sum_{q\in\mathbb{Z}^\nu} e^{-\frac{q^2}{4t}} u_q(t), \tag{1.4}$$

$$\text{with} \quad u_q(t) = d + a_{1,q} t + \cdots + a_{r-1,q} t^{r-1} + \cdots, q \in \mathbb{Z}^\nu. \tag{1.5}$$

In (1.5), each expansion is Borel summable and u_q denotes the Borel sum of such an expansion. Since $a_{1,0} = a_1$, $a_{2,0} = a_2$, $\ldots$, the expansion in (1.3) is Borel summable but (1.4) shows us that the knowledge of the coefficients $a_1, a_2 \ldots$ does not allow one to recover the left hand side of (1.3) by Borel summation.

Let us now state more precisely the assumptions on V. Let $\alpha \in \mathbb{R}$ and let μ be a Borel measure on $\mathbb{R}^\nu$ with values in some complex finite dimensional space of square matrices, such that, for some $\varepsilon > 0$

$$\int_{\mathbb{R}^\nu} \exp(\varepsilon \xi^2) d|\mu|(\xi) < +\infty. \tag{1.6}$$

We suppose that $V(x) = \alpha x^2 - c(x)$[1] where

$$c(x) = \int_{\mathbb{R}^\nu} \exp(ix\cdot\xi) d\mu(\xi).$$

c must be viewed as a perturbation. Assuming c is complex valued instead of matrix valued does not simplify the method and does not change the results. In particular, (1.6) implies that V is analytic on $\mathbb{C}^\nu$ and c is bounded on $\mathbb{R}^\nu$. In fact, our results hold if αx^2 is replaced by an arbitrary quadratic form on $\mathbb{R}^\nu$. For the sake of simplicity, we choose not to write the proofs in this case (see [Ha6] for a generalization of the deformation formula). Note also that functions c such that $c(x) := \exp(ix_1)$ or $c(x) := \exp(-x^2)$ satisfy our assumptions.

Quantum mechanics gives many examples of divergent expansions. We focus on semi-classical expansions related to the Schrödinger equation since they present a lot of similarities with the small time expansion of the heat kernel. The semi-classical viewpoint allows to expand a quantum quantity in terms of powers of h. The coefficients of this expansion can be viewed as classical quantities. Giving a meaning to the sum of this expansion by using only the coefficients allows one to recover the quantum quantity by means of classical quantities [B-B, V1, V2]. The same interpretation holds for the expansion of the heat kernel for small t (the parameter t may be viewed as β, the inverse of the temperature). The heat kernel of an operator can be viewed as a quantum

[1]The minus sign in front of $c(x)$ is chosen for future convenience.

quantity: for instance, its trace, if it exists, gives the partition function of the spectrum of the operator. The coefficients of the expansion have a classical interpretation: for instance in (1.1), the main term in the exponential is the square of the distance between x and y, which is a classical quantity. See also Remark 8.4 in [Ha3].

Recovering quantum quantities with the help of the coefficients of their semi-classical expansion is a question considered by Voros [V1, V2] and Delabaere, Dillinger, Pham [D-D-P] for the one dimensional Schrödinger equation. Their use of Borel summation is not elementary as ours. Their assumptions, which allows to consider tunnelling for instance, involve to deal with ramified singularities in the Borel plane (Ecalle's alien calculus). In a following paper [Ha8], we prove that the h expansion of the partition function of the Schrödinger operator is Borel summable in all dimensions but with restrictive assumptions on the potential allowing a simple Borel summation process.

Concerning the expansion of the heat kernel for small t, we are not aware of references using Borel summation. However, Colin de Verdière [Co1, Co2] gives a Poisson formula in the case of a smooth Riemannian manifold. The setting of his work is much more general than ours, but the result is asymptotic and does not give an exact formula. The exact formula on the torus is a direct consequence of the result on $\mathbb{R}^\nu$: it is just putting together independent expansions (one more time, we do not need alien calculus). The case of the torus is very simple but singular.

It is convenient to write the potential c as the Fourier transform of a Borel measure. This point of view is used by many authors [It, Ga, A-H], working with a rigorous definition of Wiener and Feynman integrals. The assumptions on c and the method allow to consider in a natural way $p_t(x, y)$ with $x, y \in \mathbb{C}^\nu, t \in \mathbb{C}, \mathcal{R}et \geqslant 0$ instead of $x, y \in \mathbb{R}^\nu, t \in \mathbb{R} \cup i\mathbb{R}$.

Our proofs use a so-called deformation formula and a so-called deformation matrix. In the free case (i.e $V(x) = \alpha x^2 - c(x)$ with $\alpha = 0$), this formula can be found in [It, Formula (77)]. In this reference, the deformation matrix is not given explicitly. One can also give a formula for the solution of the heat or Schrödinger equation with an arbitrary initial condition (of course no factorization occurs contrary to the heat kernel case), see [Ga, A-H] and more precisely [Ga, Eq.(27)] and [A-H, Eq.(3.12),(5.16)]. In these references, the deformation formula is a mean to study Feynman integral; the complex viewpoint, which is straightforward in the free case, is not considered. In a heuristic way, this formula is well known [On] and can even be considered as a particular case of a more general one [Ge-Ya]. In this setting, there is no reason to consider c as the Fourier transform of a Borel measure. See also [Fu-Os-Wi] and [Ha3]. We strongly advise the reader to look at the known heuristic proof of this formula given in the Appendix. This proof, which uses Wiener representation of the heat kernel and Wick's theorem, gives an explanation of the shape of the formula and an interpretation of a so-called deformation matrix (see also [It]) which is important in our method. However, there is a serious drawback in thinking of the deformation formula as a consequence of the existence of Wiener or Feynman integrals, in particular from a rigorous point of view. The deformation formula

is elementary: it does not use sophisticated notions about infinite-dimensional spaces. Another heuristic proof, avoiding Wiener representation, can be found in [On]. Here Wick's theorem also plays a central role. In Section 4.2, we give a rigorous proof of the deformation formula, which avoids Wiener representation but does not explain in a satisfactory way the shape of the formula. We call it deformation formula because we want to emphasize its perturbative nature. It is therefore not surprising that the assumptions (4.17) (the same as those used for the mathematical foundation of the Feynman integrals [A-H]) or (4.24) on the function c are highly restrictive. For instance, the heat kernel associated to the operator $\partial_x^2 - x^2$ can not be viewed as a perturbation of the heat kernel associated to the operator ∂_x^2 by this formula.

This formula gives an explicit solution of the heat equation viewed as a partial differential equation which is available for every $\alpha \in \mathbb{R}$ and small $t \in \mathbb{C}$ with $\mathcal{R}et > 0$. For this purpose some properties of the deformation matrix for these values of t are needed (Section 4.1). At last, let us say a few words about the unicity problem. If $\alpha \leqslant 0$, the heat kernel is uniquely defined as the kernel of an analytic semi-group [Pa]. In the case $\alpha > 0$, see [Ha7].

The paper is organized as follows. In Section 2, we present some notation and recall some classical facts about Borel summation. In Section 3, we state the main results and give their proofs in Sections 4.3 and 4.4.

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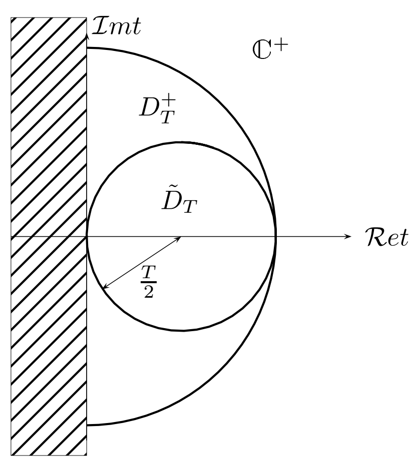
2 Preliminaries

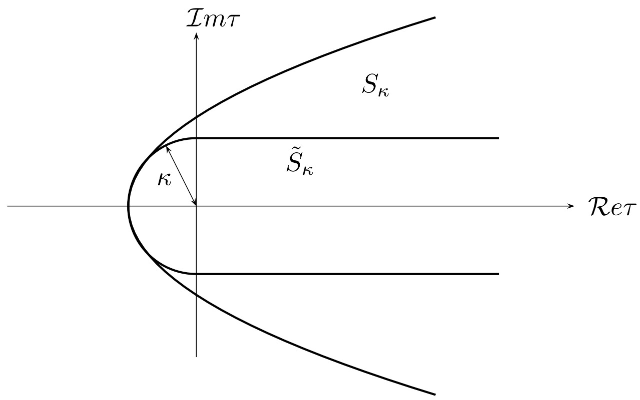
For $z = |z|e^{i\theta} \in \mathbb{C}$, $\theta \in]-\pi, \pi]$, we denote $z^{1/2} := |z|^{1/2}e^{i\theta/2}$. Let $T > 0$. Let $\mathbb{C}^+ := \{z \in \mathbb{C} | \mathcal{R}e(z) > 0\}$, $D_T := \{z \in \mathbb{C} | |z| < T\}$, $D_T^+ := D_T \cap \mathbb{C}^+$ and $\tilde{D}_T := \left\{z \in \mathbb{C} | \mathcal{R}e(\frac{1}{z}) > \frac{1}{T}\right\}$. $\tilde{D}_T$ is the open disk of center $\frac{T}{2}$ and radius $\frac{T}{2}$.

Figure 2.1:



Let $\kappa > 0$. Let $\tilde{S}_\kappa := \{z \in \mathbb{C} | d(z, [0, +\infty[) < \kappa\}$ and $S_\kappa := \{z \in \mathbb{C} || \mathcal{I}m z^{1/2}|^2 < \kappa\} = \{z \in \mathbb{C} | \mathcal{R}e z > \frac{1}{4\kappa} \mathcal{I}m^2 z - \kappa\}$. S_κ is the interior of a parabola which contains $\tilde{S}_\kappa$.

Figure 2.2:



Let Ω be an open domain in $\mathbb{C}^m$ and let F be a complex finite dimensional space. We denote by $\mathcal{A}(\Omega)$ the space of F-valued analytic functions on Ω, if there is no ambiguity on F.

Let $\mathfrak{B}$ denote the collection of all Borel sets on $\mathbb{R}^m$. An F-valued measure μ on $\mathbb{R}^m$ is an F-valued measurable function on $\mathfrak{B}$ satisfying the classical countable additivity property (cf. [Ru]). Let $|\cdot|$ be a norm on F. We denote by $|\mu|$ the positive measure defined by

$$|\mu|(E) = \sup \sum_{j=1}^{\infty} |\mu(E_j)| (E \in \mathfrak{B}),$$

the supremum being taken over all partition $\{E_j\}$ of E. In particular, $|\mu|(\mathbb{R}^m) < \infty$. Let us remark that $d\mu = h d|\mu|$ where h is some F-valued function satisfying $|h| = 1$ $|\mu|$-a.e. If f is an F-valued measurable function on $\mathbb{R}^m$ and λ is a positive measure on $\mathbb{R}^m$ such that $\int_{\mathbb{R}^m} |f| d\lambda < \infty$, one can define an F-valued measure μ by setting $d\mu = f d\lambda$. Then $d|\mu| = |f| d\lambda$.

We work with finite dimensional spaces of square matrices. We always consider multiplicative norms on these spaces ($|AB| \leqslant |A||B|$, for A and B square matrices) and we assume that $|\mathbb{1}| = 1$. For $A = (a_{i,j})_{1 \leqslant i,j \leqslant d}$ with $a_{i,j} \in \mathbb{C}$, we set $A^* = (\bar{a}_{j,i})_{1 \leqslant i,j \leqslant d}$. For $\lambda, \mu \in \mathbb{C}^\nu$, we denote $\lambda \cdot \mu := \lambda_1 \mu_1 + \cdots + \lambda_\nu \mu_\nu$, $\bar{\lambda} := (\bar{\lambda}_1, \ldots, \bar{\lambda}_\nu)$, $\mathcal{I}m\lambda := (\mathcal{I}m\lambda_1, \ldots, \mathcal{I}m\lambda_\nu)$, $\lambda^2 := \lambda \cdot \lambda$, $|\lambda| := (\lambda \cdot \bar{\lambda})^{1/2}$ (if $\lambda \in \mathbb{R}^\nu$, $|\lambda| = \sqrt{\lambda^2}$). In the whole paper, sums indexed by an empty set are, by convention, equal to zero.

Here is an improved version of a theorem of Watson. This is in fact a theorem of Nevanlinna, rediscovered by Sokal [So]. It gives a concise presentation of what

we need about Borel summation. In what follows, functions are defined on some subset of $\mathbb{C}$ and take their values in a complex finite dimensional space F. First, we need:

Definition 2.1 *Let $\kappa > 0$ and $T > 0$. We say that*

- *A function f satisfies $\mathcal{P}_{\kappa,T}$ if and only if f is analytic on $\tilde{D}_T$ (fig. 2.1) and there exist $a_0, a_1, \ldots \in F$, $R_0, R_1, \ldots$ analytic functions on $\tilde{D}_T$ such that, for every $r \geqslant 0$*[2] *and $t \in \tilde{D}_T$,*

$$f(t) = a_0 + \cdots + a_{r-1}t^{r-1} + R_r(t),$$

and for every $\bar{\kappa} < \kappa$, $\bar{T} < T$, there exists $K > 0$ such that, for every $r \geqslant 0$ and $t \in \tilde{D}_{\bar{T}}$,

$$|R_r(t)| \leqslant K \frac{r!}{\bar{\kappa}^r} |t|^r$$

- *A function*[3] *$\hat{f}$ satisfies $\mathcal{Q}_{\kappa,T}$ if and only if $\hat{f}$ is analytic on $\tilde{S}_\kappa$ (fig. 2.2) and for every $\bar{\kappa} < \kappa$, $\bar{T} < T$, there exists $K > 0$ such that, for every $\tau \in \tilde{S}_{\bar{\kappa}}$*

$$|\hat{f}(\tau)| \leqslant K e^{\frac{|\tau|}{\bar{T}}}. \tag{2.1}$$

This definition is justified by the following theorem and remark. Note that, if f satisfies $\mathcal{P}_{\kappa,T}$, the coefficients a_n are uniquely determined.

Theorem 2.2 *Let $\kappa > 0$ and $T > 0$.*

- *If f verifies $\mathcal{P}_{\kappa,T}$, then*

$$\hat{f}(\tau) := \sum_{r=0}^{\infty} \frac{a_r}{r!} \tau^r \tag{2.2}$$

admits an analytic continuation on $\tilde{S}_\kappa$ which verifies $\mathcal{Q}_{\kappa,T}$.

- *If $\hat{f}$ verifies $\mathcal{Q}_{\kappa,T}$, then*

$$f(t) := \int_0^{+\infty} \hat{f}(\tau) e^{-\frac{\tau}{t}} \frac{d\tau}{t} \tag{2.3}$$

verifies $\mathcal{P}_{\kappa,T}$.

- *$\hat{f}$ given by (2.2) is called the Borel transform of f. f given by (2.3) is called the Laplace transform of $\hat{f}$. These two transforms are inverse each to other.*

[2] If $r = 0$, the expansion must be read $f(t) = R_0(t)$, by the previous convention.

[3] In general, we denote functions defined on the Borel plane by a hat.

Remark 2.3 *If $\mathcal{P}_{\kappa,T}$ holds, the knowledge of the coefficients $a_0, a_1, \dots$ allows one to recover f: f is the Laplace transform of its Borel transform which only depends on $a_0, a_1, \dots$ by (2.2). The shape of the domain $\tilde{D}_T$ is crucial. For instance, the conclusion of this remark may fail if $\tilde{D}_T$ is replaced by a truncated cone like $C_{T,\alpha} := \{t = re^{i\theta} \in \mathbb{C} | |\theta| < \alpha, r < T\}$ with $T > 0$, $\alpha < \frac{\pi}{2}$.*

Under our assumptions on V, the conjugate heat kernel will verify at least $\mathcal{P}_{\kappa,T}$ for some $\kappa > 0$ and $T > 0$. We do not prove a resummation estimate on a truncated cone $C_{T,\alpha}$ with $T > 0$, $\alpha > \frac{\pi}{2}$ and in general, the conjugate heat kernel will not verify the assumptions of Watson's theorem (cf. [So]).

Definition 2.4 *Let $\tilde{a}_1, \dots, \tilde{a}_r, \dots \in F$. One says that the formal power series $\tilde{f} = \sum_{r \geqslant 0} \tilde{a}_r t^r$ is Borel summable if there exist $\kappa, T > 0$ and a function f satisfying $\mathcal{P}_{\kappa,T}$ such that, for every $r \geqslant 0$, $a_r = \tilde{a}_r$. f is called the Borel sum of $\tilde{f}$.*

3 Main results

The following theorem concernes the free case. In this case, it is possible to give precise properties of the Borel transform of the conjugate heat kernel. In particular, this Borel transform is analytic on the complex plane and is exponentially dominated by the square root of the Borel variable on parabolic domains which are symmetric with respect to the positive real axis.

Theorem 3.1 *Let $\varepsilon > 0$. Let μ be a measure on $\mathbb{R}^\nu$ with values in a complex finite dimensional space of square matrices verifying*

$$\int_{\mathbb{R}^\nu} \exp(\varepsilon \xi^2) d|\mu|(\xi) < \infty.$$

Let $c(x) = \int \exp(ix \cdot \xi) d\mu(\xi)$ and let u be the solution of

$$\begin{cases} \partial_t u = \partial_x^2 u + c(x) u \\ \\ u|_{t=0^+} = \delta_{x=y} \mathbb{1} \end{cases} \tag{3.1}$$

Let v be defined by $u = (4\pi t)^{-\nu/2} e^{-\frac{(x-y)^2}{4t}} v$. Then v admits a Borel transform $\hat{v}$ (with respect to t) which is analytic on $\mathbb{C}^{1+2\nu}$. Let $\kappa, R > 0$ and let

$$C := 2\Big(\int_{\mathbb{R}^\nu} \exp\big(\frac{2\kappa}{\varepsilon} + \frac{\varepsilon}{2}\xi^2 + R|\xi|\big) d|\mu|(\xi)\Big)^{1/2}.$$

Then, for every $(\tau, x, y) \in S_\kappa \times \mathbb{C}^{2\nu}$ such that $|\mathcal{I}mx| < R$ and $|\mathcal{I}my| < R$,

$$|\hat{v}(\tau, x, y)| \leqslant \exp\big(C|\tau|^{1/2}\big). \tag{3.2}$$

Remark 3.2 *S_κ is the interior of a parabola which contains $\tilde{S}_\kappa$ (fig. 2.2). Let $T > 0$. Estimate (3.2) is better than (2.1). Then $v(t,x,y)$ verifies $\mathcal{P}_{\kappa,T}$: the small time expansion of the conjugate heat kernel is Borel summable and its Borel sum is equal to v.*

The following corollary deals with the partition function on the torus and is a consequence of the Theorem 3.1.

Corollary 3.3 *Let $\varepsilon > 0$ and $d \in \mathbb{N}^*$. For each $q \in \mathbb{Z}^\nu$, let c_q be a square matrix acting on $\mathbb{C}^d$. Assume that $c_{-q} = c_q^*$ and*

$$\sum_{q\in\mathbb{Z}^\nu} e^{4\pi^2\varepsilon q^2}|c_q| < \infty.$$

Let $c(x) := \sum_{q\in\mathbb{Z}^\nu} c_q e^{2i\pi q.x}$. Let $\lambda_1 \leqslant \lambda_2 \leqslant \cdots$ be the eigenvalues of the operator $H := -\partial_x^2 - c(x)$ acting on $\mathbb{C}^d$-valued functions defined on the torus $(\mathbb{R}/\mathbb{Z})^\nu$. For each $q \in \mathbb{Z}^\nu$, there is a function $\hat{w}(q,.)$ analytic on $\mathbb{C}$ satisfying

- *For every $\kappa > 0$, there exist constants $C_1 > 0$ and $C_2 > 0$ such that, for every $q \in \mathbb{Z}^\nu$ and $\tau \in S_\kappa$,*

$$|\hat{w}(q,\tau)| \leqslant C_1 \exp\bigl(C_2|\tau|^{1/2}\bigr). \tag{3.3}$$

- *For every $t \in \mathbb{C}^+$*

$$\sum_{n=1}^{+\infty} e^{-\lambda_n t} = (4\pi t)^{-\nu/2} \sum_{q\in\mathbb{Z}^\nu} e^{-\frac{q^2}{4t}} \int_0^{+\infty} e^{-\frac{\tau}{t}} \hat{w}(q,\tau)\frac{d\tau}{t}. \tag{3.4}$$

Remark 3.4 *By the argument of Remark 3.2, Corollary 3.3 implies the following result. Let c as in Corollary 3.3. For each $q \in \mathbb{Z}^\nu$, there are numbers $a_{1,q}, a_{2,q}, \ldots \in \mathbb{C}$ and functions $R_{0,q}, R_{1,q}, \ldots \in \mathcal{A}(\mathbb{C}^+)$ such that, for every $r \geqslant 0$ and $t \in \mathbb{C}^+$,*

$$\sum_{n=1}^{+\infty} e^{-\lambda_n t} = (4\pi t)^{-\nu/2} \sum_{q\in\mathbb{Z}^\nu} e^{-\frac{q^2}{4t}} \bigl(d + a_{1,q}t + \cdots + a_{r-1,q}t^{r-1} + R_{r,q}(t)\bigr), \tag{3.5}$$

and for each $T, \kappa > 0$, there exist $K > 0$ such that

$$|R_{r,q}(t)| \leqslant K\frac{r!}{\kappa^r}|t|^r,$$

for every $r \geqslant 0, q \in \mathbb{Z}^\nu, t \in \tilde{D}_T$.

The index q in (3.5) can be viewed as labeling closed classical trajectories (geodesics) on the torus. Then q^2 denotes the length of such a geodesic. All coefficients of the expansion 3.5 have classical (or geometric) interpretation.

The following theorem deals with the harmonic case (i.e. $V(x) = \alpha x^2 - c(x)$ with $\alpha \in \mathbb{R}$) and gives a statement about the expansion of the conjugate heat kernel which provides Borel summability. But we do not obtain a statement as precise as in Theorem 3.1 about its Borel transform (the proof is established without working in the Borel plane).

Theorem 3.5 *Let $\omega \in \mathbb{R} \cup i\mathbb{R}$. Let c be as in Theorem 3.1. Let u be the solution of*

$$\begin{cases} \partial_t u = \left(\partial_x^2 - \frac{\omega^2}{4} x^2\right) u + c(x) u \\ \\ u|_{t=0^+} = \delta_{x=y} \mathbb{1} \end{cases} \tag{3.6}$$

- *Then there are a number $T > 0$, functions $a_1, a_2, \ldots \in \mathcal{A}(\mathbb{C}^{2\nu})$ and $R_0, R_1, \ldots \in \mathcal{A}(D_T^+ \times \mathbb{C}^{2\nu})$ such that*

  $$u = (4\pi t)^{-\nu/2} e^{-\frac{(x-y)^2}{4t}} \left(\mathbb{1} + a_1(x,y)t + \cdots + a_{r-1}(x,y)t^{r-1} + R_r(t,x,y)\right), \tag{3.7}$$

  for every $r \geqslant 0$, $t \in D_T^+$ and $(x,y) \in \mathbb{C}^{2\nu}$.

- *And for each $R > 0$, there exist $K > 0$ and $\kappa > 0$ such that,*

  $$|R_r(t,x,y)| \leqslant K \frac{r!}{\kappa^r} |t|^r, \tag{3.8}$$

  for every $r \geqslant 0$, $t \in D_T^+$, $(x,y) \in \mathbb{C}^{2\nu}$, $|x| < R$, $|y| < R$.

Remark 3.6

- *If $\omega \in \mathbb{R}$, the heat kernel is uniquely defined as the kernel of an analytic semi-group. In the case $\omega \in i\mathbb{R}$, one can also give a uniqueness statement (see [Ha7]) which allows one to speak about "the" heat kernel.*

- *By (3.8) and Theorem 2.2, the expansion in (3.7) admits a Borel transform satisfying $\mathcal{Q}_{\kappa,T}$. In the case $\omega = 0$, the estimate (3.8) is better than that obtained by Remark 3.2 since $\tilde{D}_T \subset D_T^+$ (fig. 2.1). In particular we get a uniform estimate when $\mathcal{R}et \to 0^+$ and $\mathcal{I}mt$ is a non-vanishing constant. Hence, we obtain information about the Schrödinger kernel. In fact, this information is contained in Theorem 3.1: it is no difficult to see that Theorem 3.1 implies the estimate (3.8) for $t \in D_T^+$.*

4 Proof of the theorems

The proof of our result uses an explicit formula of the heat kernel which has two expressions (compare (5.1) and (4.18)). This formula is based on a matrix ($\Omega^\natural$ or Ω) and a path ($q_\omega^\natural$ or q_ω). There are two cases. In the free case ($\omega = 0$), the matrix Ω is linear in t and the path q_ω does not depend on t. Consequently, the proof of our results is simple and working in the Borel plane is natural. In the

harmonic case ($\omega \neq 0$), we must deal with the t-dependence of the matrix and the path. The following subsection is devoted to the study of this matrix (let us emphasize that only Lemma 4.1, in this subsection, is useful for the study of the free case).

4.1 The deformation matrix

Let $n \in \mathbb{N}$, $\omega \in \mathbb{C}$, $t \in]0, +\infty[$ such that $|\omega t| < \pi$ and let $(s_1, \ldots, s_n) \in [0, t]$ such that $0 < s_1 < \cdots < s_n < t$. For $A \in \mathbb{C}$, denote $\operatorname{sh} A := \frac{1}{2}(e^A - e^{-A})$. In the following sections, the matrix

$$\Omega^\natural := \left(\frac{\operatorname{sh}(\omega s_{j\wedge k}) \operatorname{sh}\big(\omega(t - s_{j\vee k})\big)}{\omega \operatorname{sh}(\omega t)} \right)_{1 \leqslant j,k \leqslant n} \tag{4.1}$$

plays an important role (cf. (4.22) and (5.1)) and some of its properties must be established. However, since we shall also consider complex values of t, we study the following matrix. Assume now that $t \in \mathbb{C}$. Let $(s_1, \ldots, s_n) \in [0, 1]$ such that $0 < s_1 < \cdots < s_n < 1$. Let Ω be defined by

$$\Omega := \left(\frac{\operatorname{sh}(\omega t s_{j\wedge k}) \operatorname{sh}\big(\omega t(1 - s_{j\vee k})\big)}{\omega \operatorname{sh}(\omega t)} \right)_{1 \leqslant j,k \leqslant n}. \tag{4.2}$$

The goal of this section is to study some properties of Ω (Proposition 4.5) in particular when n, the dimension of the matrix, is large. This control, in large dimension, plays an important role in the proof of the Borel summability of the heat kernel expansion. Note that the connection of this matrix with a propagator (cf. (5.7)) seems to be important for the understanding of its properties. The following lemma gives a useful elementary property of $\bar{\Omega} := \frac{1}{t}\Omega|_{\omega=0} = \big(s_{j\wedge k}(1 - s_{j\vee k})\big)_{1\leqslant j,k\leqslant n}$ and Proposition 4.5, the goal of this subsection, can be viewed as a generalization of this lemma. For $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^{\nu n}$, let

$$\bar{\Omega} \cdot \xi \otimes_n \xi := \sum_{j,k=1}^{n} s_{j\wedge k}(1 - s_{j\vee k})\xi_j \cdot \xi_k. \tag{4.3}$$

Lemma 4.1 *For every $n \geqslant 1, (\xi_1, \ldots, \xi_n) \in \mathbb{R}^{\nu n}$ and $(s_1, \ldots, s_n) \in [0, 1]$ such that $0 < s_1 < \cdots < s_n < 1$*

$$0 \leqslant \bar{\Omega} \cdot \xi \otimes_n \xi \leqslant n \sum_{j=1}^{n} \xi_j^2.$$

Proof The upper bound of the quantity $\bar{\Omega}\cdot\xi\otimes_n\xi$ is elementary and its positivity can be viewed as a consequence of Remark 4.4. $\square$

Lemma 4.2 *Let $\tilde{T} > 0$ and $M > 0$. There exists $T > 0$ satisfying the following property. Let f be an analytic function f on $D_{\tilde{T}}$ verifying $f(0) = 0$, $f'(0) = 1$, $\sup_{t \in D_{\tilde{T}}} |f(t)| \leqslant M$ and, for every $t \in D_{\tilde{T}}$,*

$$\mathcal{R}et = 0 \Rightarrow \mathcal{R}ef(t) = 0. \tag{4.4}$$

Then, for $t \in D_T$,

$$\mathcal{R}et > 0 \Rightarrow \mathcal{R}ef(t) > 0. \tag{4.5}$$

Proof We can choose $T > 0$, depending **only** on $\tilde{T}$ and M, such that every analytic function f verifying $f(0) = 0$, $f'(0) = 1$ and $|f(t)| \leqslant M$ for $t \in D_{\tilde{T}}$ is a one-to-one analytic mapping on D_T. For small $t > 0$, $\mathcal{R}ef(t) > 0$ since $f(0) = 0$ and $f'(0) = 1$. Then by (4.4) and since f is a one-to-one analytic mapping, one gets (4.5). $\square$

Proposition 4.3 *Let $\mathcal{E}$ be the space of Borel measures $\mu = \sum_{j=1}^n \delta_{s_j} \xi_j$ such that $n \geqslant 1, \xi_j \in \mathbb{R}^\nu, s_j \in]0,1[$. Let $z \in D_\pi$. We denote by $(.,.)_z$ the following bilinear form on $\mathcal{E}$*

$$(\mu_1, \mu_2)_z := \int_0^1 \int_0^1 \frac{\operatorname{sh}(zs \wedge s') \operatorname{sh}\big(z(1 - s \vee s')\big)}{z \operatorname{sh} z} d\mu_1(s) \cdot d\mu_2(s').$$

Note that

$$(\mu_1, \mu_2)_0 = \int_0^1 \int_0^1 s \wedge s'(1 - s \vee s') d\mu_1(s) \cdot d\mu_2(s').$$

Then, for $z \in D_\pi$,

$$\forall \mu \in \mathcal{E}, |(\mu, \mu)_z| \leqslant \frac{\pi^2}{\pi^2 - |z|^2} (\mu, \mu)_0. \tag{4.6}$$

Proof For $(s, s') \in [0,1]^2$, let

$$K(s, s') := \frac{\operatorname{sh}(zs \wedge s') \operatorname{sh}\big(z(1 - s \vee s')\big)}{z \operatorname{sh} z}.$$

Then

$$\begin{cases} -\frac{d^2K}{ds^2} + z^2 K = \delta_{s=s'} \\ K|_{s=0} = K|_{s=1} = 0 \end{cases}. \tag{4.7}$$

Let $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^{\nu n}$ and $(s_1, \ldots, s_n) \in]0,1[^n$. Let u be the function on $[0,1]$ defined by

$$u(s) = \sum_{j=1}^n \frac{\operatorname{sh}(zs \wedge s_j) \operatorname{sh}\big(z(1 - s \vee s_j)\big)}{z \operatorname{sh} z} \xi_j.$$

u is continuous and piecewise differentiable on $[0,1]$. Let $\mu := \sum_{j=1}^n \delta_{s_j}\xi_j$. By (4.7)

$$\begin{cases} -\frac{d^2u}{ds^2} + z^2 u = \mu \\ \\ u(0) = u(1) = 0 \end{cases} . \tag{4.8}$$

For $n \geqslant 1$, $k, l \in \{1, \dots, \nu\}$ and $s \in [0,1]$, let $e_{n,k,l}(s) := \sqrt{2}\sin(n\pi s)\delta_{k=l}$. $(e_{n,k})_{n,k}$ is an orthonormal basis of $L^2([0,1], \mathbb{C}^\nu)$ which diagonalizes the unbounded operator $S := -\frac{d^2}{ds^2}$. Let

$$\begin{aligned} H_0^1 := & \quad \Big\{ f \in L^2([0,1], \mathbb{C}^\nu) \Big| \frac{df}{ds} \in L^2, f(0) = f(1) = 0 \Big\} \\ = & \quad \Big\{ \sum_{n,k} f_{n,k} e_{n,k} \Big| \sum_{n,k} |n f_{n,k}|^2 < \infty \Big\}, \end{aligned}$$

$$H^{-1} := \left\{ \sum_{n,k} f_{n,k} e_{n,k} \middle| \sum_{n,k} \left| \frac{f_{n,k}}{n} \right|^2 < \infty \right\}.$$

For $(f,g) \in H^{-1} \times H_0^1$, let

$$\langle f, g \rangle := \int_0^1 \bar{f}(s) \cdot g(s) ds = \sum_{n,k} \bar{f}_{n,k} g_{n,k}.$$

Note that S can be viewed as an isomorphism from H_0^1 to H^{-1}. The function u belongs to H_0^1 and by (4.8), $\mu = (S + z^2)u \in H^{-1}$. Moreover, $(\mu,\mu)_z = \int_0^1 u \cdot d\mu$ and π^2 is the lowest eigenvalue of S (every eigenvalue of S are real). Hence, for $z \in D_\pi$,

$$(\mu,\mu)_z = \left\langle \mu, (S+z^2)^{-1}\mu \right\rangle, (\mu,\mu)_0 = \left\langle \mu, S^{-1}\mu \right\rangle \tag{4.9}$$

For $\lambda \in [\pi^2, +\infty[$ and $z \in D_\pi$, the following inequality holds

$$|(\lambda + z^2)^{-1}| \leqslant \frac{\pi^2}{\pi^2 - |z|^2} \times \lambda^{-1} \tag{4.10}$$

Then (4.6) is a consequence of (4.9) and (4.10). □

Let $v(s) := \sum_{j=1}^n s \wedge s_j (1 - s \vee s_j)\xi_j$. Then

$$(\mu,\mu)_0 = \int_0^1 v \cdot d\mu = \int_0^1 \left(\frac{dv}{ds} \right)^2$$

since $\mu = -\frac{d^2v}{ds^2}$. One gets

Remark 4.4 $(.,.)_0$ *is a positive definite symmetric bilinear form on* $\mathcal{E}$.

For $(\xi_1, \dots, \xi_n) \in \mathbb{R}^{\nu n}$ and $0 < s_1 < \dots < s_n < 1$, let

$$\Omega.\xi \otimes_n \xi := \sum_{j,k=1}^n \frac{\mathrm{sh}(\omega t s_{j \wedge k}) \, \mathrm{sh}\big(\omega t (1 - s_{j \vee k})\big)}{\omega \, \mathrm{sh}(\omega t)} \xi_j \cdot \xi_k. \tag{4.11}$$

Proposition 4.5 *Let $\omega \in \mathbb{R} \cup i\mathbb{R}$. There exists $T_d > 0$ such that for every $n \geqslant 1, (s_1, \dots, s_n) \in [0,1]^n, (\xi_1, \dots, \xi_n) \in \mathbb{R}^{\nu n}, t \in \mathbb{C}$ with the condition $0 < s_1 < \cdots < s_n < 1, |t| < T_d$,*

$$\mathcal{R}et \geqslant 0 \Rightarrow \mathcal{R}e\,(\Omega.\xi \otimes_n \xi) \geqslant 0, \tag{4.12}$$

$$|\Omega.\xi \otimes_n \xi| \leqslant 2n|t| \sum_{j=1}^{n} \xi_j^2. \tag{4.13}$$

Proof Let $\mu := \sum_{j=1}^n \delta_{s_j} \xi_j$. Then $\Omega.\xi \otimes_n \xi = t(\mu,\mu)_{\omega t}$ and $\bar{\Omega} \cdot \xi \otimes_n \xi = (\mu,\mu)_0$. Let $\tilde{T} := \frac{\pi}{\sqrt{2}|\omega|}$. By (4.6) and Lemma 4.1, (4.13) holds for $t \in D_{\tilde{T}}$. Note that (4.13) can also be obtained directly without using (4.6).

Let us choose arbitrary $\xi_1, \dots, \xi_n$ such that $(\xi_1, \dots, \xi_n)$ does not vanish. By Remark 4.4, $(\mu,\mu)_0 \neq 0$. Let f be the function defined by

$$f(t) = t\frac{(\mu,\mu)_{\omega t}}{(\mu,\mu)_0} = \frac{\Omega.\xi \otimes_n \xi}{(\mu,\mu)_0}.$$

By (4.6), f is bounded on $D_{\tilde{T}}$ by $2 \times \frac{\pi}{\sqrt{2}|\omega|}$. Since f is odd and $\omega \in \mathbb{R} \cup i\mathbb{R}$, $\overline{f(t)} = f(\bar{t})$, for $t \in D_{\tilde{T}}$. Then $f(t) = t + at^3 + bt^5 + \cdots$ with $a, b, \dots$ real and (4.4) holds. Obviously, $f(0) = 0$ and $f'(0) = 1$. Then we can use Lemma 4.2 and there exists $T_d < \tilde{T}$ such that $\mathcal{R}ef(t) > 0$ for $t \in D^+_{T_d}$. Since $(\mu,\mu)_0 \in]0,+\infty[$, (4.12) holds for $t \in D^+_{T_d}$. □

4.2 The deformation formula

Let $(x,y) \in \mathbb{C}^{2\nu}$, $t \in \mathbb{C}^*$ and $\omega \in \mathbb{R} \cup i\mathbb{R}$. Let u_ω be defined by

$$u_\omega = \Big(4\pi \frac{\mathrm{sh}(\omega t)}{\omega}\Big)^{-\nu/2} \exp\Big(-\frac{1}{4}\frac{\omega}{\mathrm{sh}(\omega t)}(\mathrm{ch}(\omega t)(x^2+y^2) - 2x \cdot y)\Big). \tag{4.14}$$

Then, by one variant of Mehler's formula,

$$\partial_t u_\omega = \big(\partial_x^2 - \frac{\omega^2}{4}x^2\big)u_\omega, u_\omega|_{t=0^+} = \delta_{x=y}.$$

Note that

$$u_0 = (4\pi t)^{-\nu/2}\, e^{-\frac{(x-y)^2}{4t}}.$$

We denote by $q^\natural_\omega$ the extremal path of the action $S := \int_0^t \big(\dot{q}^2(s) + \omega^2 q^2(s)\big) ds$ such that $q^\natural_\omega(0) = y$ and $q^\natural_\omega(t) = x$. Then

$$q^\natural_\omega(s) = \frac{1}{\mathrm{sh}(\omega t)}\Big(\mathrm{sh}(\omega s)x + \mathrm{sh}\big(\omega(t-s)\big)y\Big). \tag{4.15}$$

Let $(\xi_1, \dots, \xi_n) \in \mathbb{R}^{\nu n}$. Let us define, for $s \in [0,1]$,

$$q_\omega(s) := \frac{1}{\mathrm{sh}(\omega t)}\Big(\mathrm{sh}(\omega t s)x + \mathrm{sh}\big(\omega t(1-s)\big)y\Big)$$

and, for $s = (s_1, \dots, s_n) \in [0,1]^n$,

$$q^n_\omega(s) \cdot \xi := q_\omega(s_1) \cdot \xi_1 + \cdots + q_\omega(s_n) \cdot \xi_n.$$

Remark 4.6 *There exists $\delta > 0$ such that, for every $(t, \omega) \in \mathbb{C}^2$, $s \in [0,1]$, $(x, y) \in \mathbb{C}^{2\nu}$,*

$$|q_\omega(s)| \leqslant 4\max(|x|, |y|) \mathit{if}\ |\omega t| < \delta. \tag{4.16}$$

Proposition 4.7 *Let μ be a measure on $\mathbb{R}^\nu$ with values in a complex finite dimensional space of square matrices. Let us assume that for every $R > 0$*

$$\int_{\mathbb{R}^\nu} \exp(R|\xi|) d|\mu|(\xi) < \infty. \tag{4.17}$$

Let $c(x) := \int_{\mathbb{R}^\nu} \exp(ix \cdot \xi) d\mu(\xi)$. Let $\omega \in \mathbb{R} \cap i\mathbb{R}$. Let v be defined by

$$\begin{cases} v = \mathbb{1} + \sum_{n \geqslant 1} v_n, \\ \\ v_n(t, x, y) = t^n \int_{0 < s_1 < \cdots < s_n < 1} \int_{\mathbb{R}^{\nu n}} e^{-\Omega.\xi \otimes_n \xi} e^{i q^n_\omega(s) \cdot \xi} d^{\nu n} \mu^{\otimes}(\xi) d^n s. \end{cases} \tag{4.18}$$

In (4.18), $d^n s$ denotes $ds_1 \cdots ds_n$ and $d^{\nu n}\mu^{\otimes}(\xi)$ denotes $d\mu(\xi_n) \cdots d\mu(\xi_1)$.

Then there exists $T_c > 0$ such that $v \in \mathcal{A}(D^+_{T_c} \times \mathbb{C}^{2\nu})$ and the function $u := u_\omega v$ is solution of (3.6). If $\omega = 0$, $v \in \mathcal{A}(\mathbb{C}^+ \times \mathbb{C}^{2\nu})$.

Proof Let T_d given by Proposition 4.5 and let δ given by Remark 4.6. We choose

$$T_c = \min\Big(T_d, \frac{\delta}{|\omega|}\Big). \tag{4.19}$$

Let $R > 0$ and let

$$A := \int \exp(4R|\xi|) d|\mu|(\xi).$$

Let $(x, y) \in \mathbb{C}^{2\nu}$ such that $|x|, |y| < R$. Let $t \in D^+_{T_c}$. By (4.12) and (4.16)

$$|\exp(-\Omega.\xi \otimes_n \xi) \exp(i q^n_\omega(s) \cdot \xi)| \leqslant \exp\big(4R(|\xi_1| + \cdots + |\xi_n|)\big). \tag{4.20}$$

Hence

$$|v_n| \leqslant \frac{(|t|A)^n}{n!},$$

so the series in (4.18) converges absolutely. Since R is arbitrary, v is well defined on $D^+_{T_c} \times \mathbb{C}^{2\nu}$. By dominated convergence theorem, one can check that v_n and hence $v \in \mathcal{A}(D^+_{T_c} \times \mathbb{C}^{2\nu})$.

Let us verify that the function $u := u_\omega v$, with v given by (4.18), is solution of (3.6). A solution u of (3.6) is given, if we use the relation $u = u_\omega v$ by a solution v of the conjugate equation

$$\begin{cases} \Big(\partial_t - \frac{2}{u_\omega} \partial_x u_\omega \cdot \partial_x\Big) v = \partial_x^2 v + c(x) v \quad (t \neq 0) \\ \\ v|_{t=0^+} = \mathbb{1} \end{cases} .$$

Notice that

$$-\frac{2}{u_\omega}\partial_x u_\omega = \frac{\omega}{\operatorname{sh}(\omega t)}\big(\operatorname{ch}(\omega t)x - y\big) = \dot{q}^\natural_\omega(t).$$

Set $v_0 := \mathbb{1}$. It is then sufficient to verify that v_n given by (4.18) satisfies

$$\begin{cases} \big(\partial_t + \dot{q}^\natural_\omega(t)\cdot\partial_x\big)v_n = \partial_x^2 v_n + c(x)v_{n-1} \\ \\ v_n|_{t=0^+} = 0 \end{cases}, \tag{4.21}$$

for $(t,x,y) \in D^+_{T_c} \times \mathbb{C}^{2\nu}$, $n \geqslant 1$.

It suffices to check (4.21) for $(t,x,y) \in]0,T_c[\times\mathbb{C}^{2\nu}$. By (4.18) and the definition of c

$$v_n = \int_{0<s_1<\cdots<s_n<t} F_n d^n s,$$

where

$$F_n := \Big[\exp\big(\sum_{j,k=1}^{n} \Omega^\natural_{j,k}\partial_{z_j}\cdot\partial_{z_k}\big)c(z_n)\cdots c(z_1)\Big]\Big|_{\substack{z_1 = q^\natural_\omega(s_1) \\ \cdots \\ z_n = q^\natural_\omega(s_n)}}. \tag{4.22}$$

Here $\Omega^\natural$ denotes the $n\times n$ matrix defined by (4.1). In the Appendix, we give an heuristic explanation of this result. Here is a rigorous verification. We have

$(\partial_t + \dot{q}^\natural_\omega(t)\cdot\partial_x)v_n =$ (boundary) + (interior) where

$$(\text{boundary}) := \int_{0<s_1<\cdots<s_{n-1}<t} F_n|_{s_n=t} d^{n-1}s,$$

$$(\text{interior}) := \int_{0<s_1<\cdots<s_n<t} \big(\partial_t + \dot{q}^\natural_\omega(t)\cdot\partial_x\big)F_n d^n s.$$

In particular, $\Omega^\natural_{j,k}|_{s_n=t} = 0$ for $j=n$ or $k=n$. Therefore

$$(\text{boundary}) = c(x)v_{n-1}.$$

Now we want to prove that (interior) $= \partial_x^2 v_n$.

Using, for instance, explicit expressions of $\dot{q}^\natural_\omega(t)$ and $q^\natural_\omega$, we get, for $s \in]0,t[$ and $m = 1,\dots,\nu$,

$$\big(\partial_t + \dot{q}^\natural_\omega(t)\cdot\partial_x\big)q^\natural_{\omega,m}(s) = 0. \tag{4.23}$$

Then, if $\varphi(z_1,\dots,z_n)$ is an arbitrary differentiable function of $(z_1,\dots,z_n) \in \mathbb{C}^{\nu n}$,

$$\big(\partial_t + \dot{q}^\natural_\omega(t)\cdot\partial_x\big)[\varphi(z_1,\dots,z_n)]|_{\substack{z_1 = q^\natural_\omega(s_1) \\ \cdots \\ z_n = q^\natural_\omega(s_n)}} = 0.$$

So, since $\partial_t\Omega^\natural_{j,k} = \frac{\operatorname{sh}(\omega s_j)}{\operatorname{sh}(\omega t)}\frac{\operatorname{sh}(\omega s_k)}{\operatorname{sh}(\omega t)}$, denoting $B_{j,k} := \frac{\operatorname{sh}(\omega s_j)}{\operatorname{sh}(\omega t)}\frac{\operatorname{sh}(\omega s_k)}{\operatorname{sh}(\omega t)}$,

$$(\text{interior}) = \int_{0<s_1<\cdots<s_n<t} G_n d^n s,$$

where

$$G_n := \Big[\Big(\sum_{j,k=1}^{n} B_{j,k}\partial_{z_j}\cdot\partial_{z_k}\Big)\exp\Big(\sum_{j,k=1}^{n}\Omega^{\natural}_{j,k}\partial_{z_j}\cdot\partial_{z_k}\Big)c(z_n)\cdots c(z_1)\Big]\Big|_{\substack{z_1=q^{\natural}_\omega(s_1)\\ \cdots\\ z_n=q^{\natural}_\omega(s_n)}}.$$

For $(\alpha,\beta)\in\{1,\dots,\nu\}^2$, one has $\partial_{x_\beta}q^{\natural}_{\omega,\alpha}=\delta_{\alpha=\beta}\times\frac{\mathrm{sh}(\omega s)}{\mathrm{sh}(\omega t)}$. Since $q^{\natural}_\omega$ is linear with respect to x,

$$\partial_x^2[c(z_n)\cdots c(z_1)]|_{\substack{z_1=q^{\natural}_\omega(s_1)\\ \cdots\\ z_n=q^{\natural}_\omega(s_n)}} = \Big[\Big(\sum_{j,k=1}^{n} B_{j,k}\partial_{z_j}\cdot\partial_{z_k}\Big)c(z_n)\cdots c(z_1)\Big]\Big|_{\substack{z_1=q^{\natural}_\omega(s_1)\\ \cdots\\ z_n=q^{\natural}_\omega(s_n)}}.$$

Then $\partial_x^2F_n=G_n$. Hence

$$(\text{interior})=\partial_x^2 v_n.$$

This proves that the function $u:=u_\omega v$ is solution of (3.6).

In the case $\omega=0$, $\Omega.\xi\otimes_n\xi=t\bar{\Omega}\cdot\xi\otimes_n\xi$ and $q_\omega(s)=y+\frac{s}{t}(x-y)$. We can get directly that $v\in\mathcal{A}(\mathbb{C}^+\times\mathbb{C}^{2\nu})$ and that $u=u_0v$ is solution of (3.6). $\square$

Remark 4.8 *Weaker assumptions than (4.17) can also be considered. One can assume that, for every $m\geqslant 0$,*

$$\int_{\mathbb{R}^\nu}|\xi|^m d|\mu|(\xi)<\infty. \tag{4.24}$$

Then, for instance in the free case, $v\in\mathcal{C}^\infty\big(\overline{\mathbb{C}^+}\times\mathbb{R}^{2\nu}\big)\cap\mathcal{A}\big(\mathbb{C}^+,\mathcal{C}^\infty(\mathbb{R}^{2\nu})\big)$.

4.3 Proof of Theorem 3.1 and Corollary 3.3

In the study of Borel transform of the conjugate heat kernel in the free case, the Borel transform of $t\longmapsto t^n\exp(-Bt)$, $B\in\mathbb{C}$, which is related to Bessel functions, plays a specific role.

Lemma 4.9 *For every $z\in\mathbb{C}$, let us denote*

$$J(z):=\sum_{n\geqslant 0}(-1)^n\frac{z^n}{(n!)^2}=\int_0^\pi\cos\big(2z^{1/2}\sin(\varphi)\big)\frac{d\varphi}{\pi}.$$

The following estimate holds for every $z\in\mathbb{C}$

$$|J(z)|\leqslant\exp\big(2|\mathcal{I}m z^{1/2}|\big). \tag{4.25}$$

These properties are easy to prove and well known since $J(z)=J_0\big(2z^{1/2}\big)$, J_0 denoting the Bessel function of order 0.

Lemma 4.10 *Let $n \geqslant 1$. For $B \in \mathbb{C}$, let $\tau \longmapsto K_n(B,\tau)$ be the Borel transform of the function $t \longmapsto t^n \exp(-Bt)$. Then, for $\tau \in \mathbb{C}$,*

$$K_n(B,\tau) = \tau^n \sum_{m\geqslant 0} \frac{(-1)^m}{m!(m+n)!}(B\tau)^m = \tau^n \int_0^1 \frac{(1-\theta)^{n-1}}{(n-1)!} J(\theta B\tau) d\theta. \quad (4.26)$$

$$|K_n(B,\tau)| \leqslant \frac{|\tau|^n}{n!} \exp\Big(2\sqrt{B}|\mathcal{I}m(\tau^{1/2})|\Big) \text{ for } B \geqslant 0. \quad (4.27)$$

Proof Since $t^n \exp(-Bt) = \sum_{m\geqslant 0}(-1)^m \frac{t^{n+m}}{m!} B^m$, using the definition of the Borel transform, we get the first equality of (4.26). Since

$$J(\theta B\tau) = \sum_{m\geqslant 0} (-1)^m \frac{(\theta B\tau)^m}{(m!)^2},$$

the second equality of (4.26) is a consequence of $\int_0^1 \frac{(1-\theta)^{n-1}}{(n-1)!} \frac{\theta^m}{m!} d\theta = \frac{1}{(m+n)!}$. (4.27) is a consequence of (4.25) and (4.26). $\Box$

Now we prove Theorem 3.1. By Proposition (4.7) with $\omega = 0$, the function $u := u_0 v$ is solution of (3.1) where $v := \mathbb{1} + \sum_{n\geqslant 1} v_n$,

$$v_n(t,x,y) := \int_{0<s_1<\cdots<s_n<1} \int_{\mathbb{R}^{\nu n}} G_n d^{\nu n}\mu^{\otimes}(\xi) d^n s, \quad (4.28)$$

and

$$G_n := t^n \exp(-t\bar{\Omega}\cdot \xi \otimes_n \xi) \exp(iq_0^n(s)\cdot \xi).$$

Here $q_0^n(s)\cdot\xi := q_0(s_1)\cdot\xi_1 + \cdots + q_0(s_n)\cdot \xi_n$ where $q_0(s) := y + s(x-y)$. Since $q_0^n(s)\cdot\xi$ does not depend on t, Lemma 4.10 will allow us to obtain a convenient formulation of the Borel transform of v.

Let $\hat{F}_n, \hat{v}_n, \hat{v}$ be defined by

$$\hat{F}_n := \exp(iq_0^n(s)\cdot\xi) K_n(\bar{\Omega}\cdot\xi\otimes_n \xi, \tau), \quad (4.29)$$

$$\hat{v}_n(\tau,x,y) := \int_{0<s_1<\cdots<s_n<1} \int_{\mathbb{R}^{\nu n}} \hat{F}_n d^{\nu n}\mu^{\otimes}(\xi) d^n s, \quad (4.30)$$

$$\hat{v} := \mathbb{1} + \sum_{n\geqslant 1} \hat{v}_n. \quad (4.31)$$

Let $\kappa, R > 0$. Let $x, y \in \mathbb{C}^\nu$ be such that $|\mathcal{I}mx|, |\mathcal{I}my| < R$ and let $\tau \in S_\kappa$. Let us check that the integral giving the definition of $\hat{v}_n$ is absolutely convergent and hence that the series giving the definition of $\hat{v}$ is absolutely convergent. By (4.27)

$$|\hat{F}_n| \leqslant \exp\big(R(|\xi_1| + \cdots + |\xi_n|)\big) \times \frac{|\tau|^n}{n!} \exp\Big(2\sqrt{\bar{\Omega}\cdot\xi\otimes_n \xi}|\mathcal{I}m(\tau^{1/2})|\Big).$$

By Lemma 4.1

$$\begin{aligned}
2|\mathcal{I}m(\tau^{1/2})|\sqrt{\bar{\Omega}\cdot\xi\otimes_n\xi} \leqslant\quad & 2\kappa^{1/2}\sqrt{n(\xi_1^2+\cdots+\xi_n^2)} \\
\leqslant\quad & 2\times\Big(\frac{2\kappa n}{\varepsilon}\Big)^{1/2}\times\sqrt{\frac{\varepsilon}{2}(\xi_1^2+\cdots+\xi_n^2)} \\
\leqslant\quad & \frac{2\kappa n}{\varepsilon}+\frac{\varepsilon}{2}(\xi_1^2+\cdots+\xi_n^2).
\end{aligned}$$

Hence

$$|\hat{F}_n| \leqslant \exp\big(R(|\xi_1|+\cdots+|\xi_n|)\big)\times\frac{|\tau|^n}{n!}\exp\big(\frac{2\kappa n}{\varepsilon}\big)\exp\Big(\frac{\varepsilon}{2}(\xi_1^2+\cdots+\xi_n^2)\Big). \quad (4.32)$$

Let

$$A := \int \exp\big(\frac{2\kappa}{\varepsilon}+\frac{\varepsilon}{2}\xi^2+R|\xi|\big)d|\mu|(\xi).$$

The integral in (4.30) is absolutely convergent and

$$|\hat{v}_n| \leqslant \frac{1}{(n!)^2}\times(A|\tau|)^n.$$

Then the series in (4.31) is absolutely convergent. Since $|\hat{v}| \leqslant \sum_{n\geqslant 0}\frac{1}{(n!)^2}\times(A|\tau|)^n = J(-A|\tau|)$ and by (4.25)

$$|\hat{v}| \leqslant \exp\big(2(A|\tau|)^{1/2}\big).$$

This proves (3.2).

By dominated convergence theorem, one can check that $\hat{v}_n$ and hence $\hat{v}$ are analytic on $S_\kappa\times\mathbb{C}^{2\nu}$ and hence on $\mathbb{C}^{1+2\nu}$ since κ is arbitrary.

Let us prove that v is the Laplace transform of $\hat{v}$. Let $t\in]0,+\infty[$. By (4.29) and since $t\longmapsto t^n\exp(-Bt)$ is the Laplace transform of $\tau\longmapsto K_n(B,\tau)$,

$$t^n\exp(iq_0^n(s)\cdot\xi)\exp(-t\bar{\Omega}\cdot\xi\otimes_n\xi) = \int_0^{+\infty}\hat{F}_n(\tau)e^{-\frac{\tau}{t}}\frac{d\tau}{t}.$$

Then, by (4.28), v is the Laplace transform of $\hat{v}$.

Now we may prove Corollary 3.3. Let $p_t(x,y)$ be the solution of

$$\begin{cases} \partial_t p_t(x,y) = \big(\partial_x^2+c(x)\big)p_t(x,y) \\ p|_{t=0}(x,y) = \sum_{q\in\mathbb{Z}^\nu}\delta_{x=y+q}\mathbb{1} \end{cases} \quad (4.33)$$

Since the torus is compact and c is a Hermitian matrix-valued analytic function, the spectrum of H is real, discret and, for $t\in]0,+\infty[$,

$$\sum_{n=1}^{+\infty}e^{-\lambda_n t} = \int_{[0,1]^\nu}\mathrm{Tr}\big(p_t(x,x)\big)dx.$$

Let $\mu := \sum_{q\in\mathbb{Z}^\nu} c_q \delta_{\xi=2\pi q}$. Then c satisfies the assumptions of Theorem 3.1. Let $\hat{v}$ and C be defined as in Theorem 3.1. Then

$$p_t(x,y) = (4\pi t)^{-\nu/2} \sum_{q\in\mathbb{Z}^\nu} \exp\Big(-\frac{(x-y-q)^2}{4t}\Big) \int_0^{+\infty} \hat{v}(\tau,x,y+q) e^{-\frac{\tau}{t}} \frac{d\tau}{t}. \tag{4.34}$$

Let $\kappa > 0$. By (3.2), the quantity $\hat{v}(\tau,x,y+q)$ is bounded (uniformly in $q \in \mathbb{Z}^\nu$) by $\exp\big(C|\tau|^{1/2}\big)$ and C only depends on $\max\big(|\mathcal{I}mx|, |\mathcal{I}my|\big)$ and κ. Hence the series in (4.34) is convergent for $t \in \mathbb{C}^+$, $x \in \mathbb{C}^\nu$ and $y \in \mathbb{C}^\nu$. Let

$$\hat{w}(q,\tau) := \int_{[0,1]^\nu} \mathrm{Tr}\big(\hat{v}(\tau,x,x+q)\big) dx,$$

$$C_1 := \sup\Big\{\frac{|\operatorname{Tr} M|}{|M|} \big| M \neq 0\Big\}, C_2 := 2\Big(\int \exp\big(\frac{2\kappa}{\varepsilon} + \frac{\varepsilon}{2}\xi^2\big) d|\mu|(\xi)\Big)^{1/2}.$$

Here the supremum is taken over non-vanishing $d \times d$ complex matrices. Then (3.4) and (3.3) hold.

4.4 Proof of Theorem 3.5

The following lemma will be useful for the proof of Theorem 3.5.

Lemma 4.11 *Let $T > 0$, $K_1 > 0$, and $\sigma_1 > 0$. Let f be an analytic function on D_T^+. Assume that, for every $r \geqslant 0$, there exist f_r analytic on D_T and g_r analytic on D_T^+ such that*

$$f = f_r + g_r,$$
$$|f_r(t)| \leqslant K_1 \sigma_1^r r! \text{ for every } t \in D_T,$$
$$|g_r(t)| \leqslant K_1 \sigma_1^r r! |t|^r \text{ for every } t \in D_T^+.$$

Let $\sigma_2 > \sigma_1$. Then there exist $K_2 > 0$, $a_0, a_1, \ldots \in \mathbb{C}$, $R_0, R_1, \ldots \in \mathcal{A}(D_T^+)$ such that

$$\begin{cases} f(t) = a_0 + \cdots + a_{r-1}t^{r-1} + R_r(t) \\ \\ |R_r(t)| \leqslant K_2 \sigma_2^r r! |t|^r \end{cases}, \tag{4.35}$$

for every $r \geqslant 0$ and $t \in D_T^+$.

Proof Let $r \geqslant 0$. Let M:=$\sup_{|t|<T} |f_r(t)|$ and $a_0 := f_r(0), \ldots, a_{r-1} := \frac{1}{(r-1)!} f_r^{(r-1)}(0)$. Let w_r be the analytic function on D_T defined by

$$w_r(t) = \frac{1}{t^r}\left(f_r(t) - (a_0 + \cdots + a_{r-1}t^{r-1})\right).$$

Let $q = 0, \ldots, r-1$ and let $t \in D_T$. By Cauchy formula, $|a_q t^q| \leqslant M$. By maximum modulus principle

$$|w_r(t)| \leqslant \frac{M}{T^r}(1+r) \leqslant \frac{K_1 \sigma_1^r r!}{T^r}(1+r).$$

Let $\sigma_2 > \sigma_1$. Let us choose $K_2 > 0$ such that, for $r \geqslant 0$, $\frac{K_1\sigma_1^r}{T^r}(1+r) \leqslant \frac{1}{2}K_2\sigma_2^r$ and $K_1\sigma_1^r \leqslant \frac{1}{2}K_2\sigma_2^r$. Let $R_r(t) := t^r w_r(t) + g_r(t)$. Then (4.35) holds for $t \in D_T^+$. $\square$

Remark 4.12 *In fact, we use a parametric version of Lemma 4.11. The function f may depend on additional parameters. Of course, the constants which appear in the assumptions of Lemma 4.11 must be independent of the parameters. Moreover, we have to assume that functions take their values in a complex finite dimensional space.*

Now we prove Theorem 3.5. Instead of proving (3.7) and (3.8) directly, we shall prove the existence of the following factorization

$$u = u_\omega v_\omega \tag{4.36}$$

where

$$v_\omega := \mathbb{1} + a_1^w(x,y)t + \cdots + a_{r-1}^\omega(x,y)t^{r-1} + R_r^\omega(t,x,y)$$

and R_r^ω satisfying (3.8). By (4.14), $\frac{u_\omega}{u_0}$ is analytic near $t=0$. Then (3.7) and (3.8) will hold since Borel summability properties of an expansion do not change if it is multiplied by a convergent expansion near $t=0$.

Let v and T_c be respectively defined by (4.18) and (4.19). Then v verifies all the properties of Proposition 4.7. We shall use Lemma 4.11 with Remark 4.12. For $m \geqslant 1, w \in \mathbb{C}$,

$$e^w = 1 + w + \cdots + \frac{w^{m-1}}{(m-1)!} + \frac{w^m}{(m-1)!}\int_0^1 (1-\theta)^{m-1}e^{\theta w}d\theta. \tag{4.37}$$

Let $r > 0$. Using (4.37) with $w = -\Omega.\xi \otimes_n \xi$ in (4.18), one gets, for $t \in D_{T_c}^+$, a decomposition $v = f_r + g_r$ where

$$f_r := \mathbb{1} + \sum_{\substack{n+m<r \\ n\geqslant 1, m\geqslant 0}} t^n \int_{0<s_1<\cdots<s_n<1}\int_{\mathbb{R}^{\nu n}} F_{n,m} d^{\nu n}\mu^{\otimes}(\xi)d^n s,$$

$$F_{n,m} := \frac{1}{m!}(-\Omega.\xi \otimes_n \xi)^m \exp\big(iq_\omega^n(s)\cdot\xi\big),$$

and $g_r = g_r^{(1)} + g_r^{(2)}$, where

$$g_r^{(1)} := \sum_{\substack{n+m=r \\ n\geqslant 1, m\geqslant 1}} t^n \int_{0<s_1<\cdots<s_n<1}\int_{\mathbb{R}^{\nu n}}\int_0^1 G_{n,m} d\theta d^{\nu n}\mu^{\otimes}(\xi)d^n s,$$

$$G_{n,m} := \frac{1}{(m-1)!}(-\Omega.\xi \otimes_n \xi)^m (1-\theta)^{m-1}\exp(-\theta\Omega.\xi \otimes_n \xi)\exp(iq_\omega^n(s)\cdot\xi),$$

and

$$g_r^{(2)} := \sum_{n\geqslant r} v_n.$$

In what follows, we shall fix some $R > 0$ and some $\varepsilon \in]0, \varepsilon_0[$. We denote

$$A := \int_{\mathbb{R}^\nu} \exp\big(\varepsilon\xi^2 + 4R|\xi|\big) d|\mu|(\xi)$$

and we take arbitrary $x, y \in \mathbb{C}^\nu$ such that $|x| < R$ and $|y| < R$.

We begin to check that f_r verifies assumptions of Lemma 4.11. Obviously, by definition of $\Omega.\xi \otimes_n \xi$ and $q_\omega^n(s) \cdot \xi$, f_r are analytic for $t \in D_{\frac{\pi}{|\omega|}}$. By (4.12), (4.13) and (4.16), for $t \in D_{T_c}$,

$$|f_r| \leqslant 1 + \sum_{\substack{n+m \leqslant r-1 \\ n \geqslant 1, m \geqslant 0}} \int_{0<s_1<\cdots<s_n<1} \int_{\mathbb{R}^{\nu n}} H_{n,m} d^{\nu n} |\mu|^{\otimes}(\xi) d^n s$$

where

$$H_{n,m} := \frac{T_c^n}{m!} \big(2nT_c(\sum_{j=1}^n \xi_j^2)\big)^m \exp\big(4R(|\xi_1| + \cdots |\xi_n|)\big).$$

But

$$\frac{\varepsilon^m}{m!} \big(\sum_{j=1}^n \xi_j^2\big)^m \leqslant \exp\big(\varepsilon \sum_{j=1}^n \xi_j^2\big) \leqslant \exp(\varepsilon\xi_1^2) \cdots \exp(\varepsilon\xi_n^2). \tag{4.38}$$

Then

$$|f_r| \leqslant 1 + \sum_{\substack{n+m \leqslant r-1 \\ n \geqslant 1, m \geqslant 0}} \frac{n^m}{n!} \left(\frac{2}{\varepsilon}\right)^m A^n T_c^{m+n}.$$

Let $B := \max\big(1, \frac{2}{\varepsilon}, A, T_c\big)$. Then, for $(n, m) \in \mathbb{N}^2$ such that $n + m < r, n \geqslant 1, m \geqslant 0$,

$$\frac{n^m}{n!} \left(\frac{2}{\varepsilon}\right)^m A^n T_c^{m+n} \leqslant r^r B^{2r}.$$

Since $\sum_{\substack{n+m<r \\ n \geqslant 1, m \geqslant 0}} 1 = \frac{r(r-1)}{2}$,

$$|f_r| \leqslant 1 + \frac{r(r-1)}{2} r^r B^{2r}.$$

Then Stirling formula implies the existence of $K_1 > 0$ and $\sigma_1 > 0$ such that for $t \in D_{T_c}$ and $r \geqslant 1$,

$$|f_r| \leqslant K_1 \sigma_1^r r!. \tag{4.39}$$

We check now that g_r verifies the assumptions of Lemma 4.11. Let $t \in D_{T_c}^+$. By (4.12), (4.13) and (4.16)

$$|g_r^{(1)}| \leqslant \sum_{\substack{n+m=r \\ n \geqslant 1, m \geqslant 1}} \int_{0<s_1<\cdots<s_n<1} \int_{\mathbb{R}^{\nu n}} L_{n,m} d^{\nu n} |\mu|^{\otimes}(\xi) d^n s$$

where

$$L_{n,m} := \frac{|t|^n}{m!}\big(2n|t|(\sum_{j=1}^{n}\xi_j^2)\big)^m \exp\big(4R(|\xi_1| + \cdots |\xi_n|)\big).$$

By (4.38)

$$|g_r^{(1)}| \leqslant |t|^r \sum_{\substack{n+m=r \\ n\geqslant 1, m\geqslant 1}} \frac{n^m}{n!}\left(\frac{2}{\varepsilon}\right)^m A^n.$$

By (4.20), $|v_n| \leqslant \frac{1}{n!}A^n|t|^n$. Hence

$$|g_r^{(2)}| \leqslant \frac{|t|^r}{T_c^r}\exp(AT_c).$$

Since $g_r = g_r^{(1)} + g_r^{(2)}$ and by similar arguments which allow us to obtain (4.39), we get the existence of $K_1 > 0$ and $\sigma_1 > 0$ such that for $t \in D^+_{T_c}$ and $r \geqslant 1$,

$$|g_r(t)| \leqslant K_1\sigma_1^r r!|t|^r. \tag{4.40}$$

By (4.39) and (4.40), we can use Lemma 4.11 with Remark 4.12. Then factorization (4.36) holds and R_r^ω satisfies (3.8). This proves (3.7) and (3.8).

Let us prove the regularity of the functions $a_1, a_2, \ldots$ and $R_0, R_1, \ldots$ The function V defined by $V(x) = -\frac{\omega^2}{4}x^2 + c(x)$ is analytic on $\mathbb{C}^\nu$. By a property of Minakshisundaram-Pleijel expansion , $a_1, a_2, \ldots \in \mathcal{A}(\mathbb{C}^{2\nu})$ and hence $R_0, R_1, \ldots \in \mathcal{A}(D^+_{T_c} \times \mathbb{C}^{2\nu})$.

5 Appendix

Let u be the solution of (3.6) and v be the function defined by the factorization $u = u_\omega v$. The goal of this section is to give a heuristic explanation of the following (deformation) formula

$$v = \mathbb{1} + \sum_{n\geq 1} v_n, \tag{5.1}$$

where

$$v_n := \int_{0<s_1<\cdots<s_n<t} F_n ds_1 \cdots ds_n,$$

$$F_n := \Big[\exp(\sum_{j,k=1}^{n} \Omega^\natural_{j,k}\partial_{z_j}\cdot\partial_{z_k})c(z_n)\cdots c(z_1)\Big]\Big|_{\substack{z_1 = q^\natural_\omega(s_1) \\ \cdots \\ z_n = q^\natural_\omega(s_n)}}.$$

Here $t > 0$. $\Omega^\natural$ and $q^\natural_\omega$ are defined in section 4 (cf. (4.1) and (4.15)). We prove and use this formula in a rigorous context (cf. (4.22)). This heuristic viewpoint has the advantage to explain the shape of the formula and gives a simple interpretation of matrix $\Omega^\natural$. However, from a rigorous point of view, the proof chosen in Proposition 4.7 seems to be more efficient.

For proving (5.1), we use Wiener integral representation of u and the following version of Wick's theorem. Let E be a real vector space. Let $\langle .,. \rangle$ be a scalar product on E and let A be a symmetric invertible operator. Then

$$\frac{\int_E \exp\big(\langle \theta, x\rangle\big) \exp\big(-\frac{1}{4}\langle Ax, x\rangle\big) dx}{\int_E \exp\big(-\frac{1}{4}\langle Ax, x\rangle\big) dx} = \exp\big(\langle A^{-1}\theta, \theta\rangle\big). \tag{5.2}$$

Of course, if E has finite dimension and $A > 0$, this formula is rigorous and means that the Laplace transform of a Gaussian is a Gaussian.

Let $V(x) = -\frac{\omega^2}{4}x^2 + c(x)$. The justification of the deformation formula starts from the following representation of u

$$u = \int \mathrm{Texp}\Big(\int_0^t V\big(q(s)\big)\, \mathrm{ds}\Big) \exp\Big(-\frac{1}{4}\int_0^t \dot{q}^2(s) ds\Big) dq, \tag{5.3}$$

where integration is taken over all the paths $q : \mathbb{R} \longrightarrow \mathbb{R}^\nu$ such that $q(0) = y$ and $q(t) = x$. Texp denotes the time-ordered exponential function:

$$\mathrm{Texp}\Big(\int_0^t V\big(q(s)\big)\, \mathrm{ds}\Big) := \sum_{n\geqslant 0} \int_{0<s_1<\cdots<s_n<t} V\big(q(s_n)\big) \cdots V\big(q(s_1)\big) ds_1 \cdots ds_n.$$

Then

$$u = \int \mathrm{Texp}\Big(\int_0^t c\big(q(s)\big)\, \mathrm{ds}\Big) \exp\Big(-\frac{1}{4}\int_0^t \big(\dot{q}^2(s) + \omega^2 q^2(s)\big) ds\Big) dq$$

Using the decomposition $q(s) = q_\omega^\natural(s) + w(s)$ and the fact that $q_\omega^\natural$ is an extremal path for the action

$$S(q) := \int_0^t \big(\dot{q}^2(s) + \omega^2 q^2(s)\big) ds,$$

which is quadratic with respect to q, we get $S(q) = S(q_\omega^\natural) + S(w)$. Hence

$$u = \exp\big(-\frac{1}{4}S(q_\omega^\natural)\big) \int \mathrm{Texp}\Big(\int_0^t c\big(q_\omega^\natural(s) + w(s)\big)\, \mathrm{ds}\Big) \exp\big(-\frac{1}{4}S(w)\big) dw, \tag{5.4}$$

where integration is taken over all the paths $w : \mathbb{R} \longrightarrow \mathbb{R}^\nu$ such that $w(0) = 0$ and $w(t) = 0$. By (5.4), with $c \equiv 0$,

$$u_\omega = \exp\big(-\frac{1}{4}S(q_\omega^\natural)\big) \int \exp\big(-\frac{1}{4}S(w)\big) dw.$$

Then $u = u_\omega v$ where

$$v = \frac{\int \mathrm{Texp}\Big(\int_0^t c\big(q_\omega^\natural(s) + w(s)\big)\, \mathrm{ds}\Big) \exp\big(-\frac{1}{4}S(w)\big) dw}{\int \exp\big(-\frac{1}{4}S(w)\big) dw}.$$

Then $v = \sum_{n\geqslant 0} \int_{0<s_1<\cdots<s_n<t} \tilde{v}_n ds_1 \cdots ds_n$ where

$$\tilde{v}_n := \frac{\int c\big(q_\omega^\natural(s_n) + w(s_n)\big) \cdots c\big(q_\omega^\natural(s_1) + w(s_1)\big) \exp\big(-\frac{1}{4}S(w)\big) dw}{\int \exp\big(-\frac{1}{4}S(w)\big) dw} \tag{5.5}$$

By Taylor formula

$$c\big(q^\natural_\omega(s)+w(s)\big)=[\exp(w(s)\cdot\partial_z)c(z)]|_{z=q^\natural_\omega(s)}.$$

Then

$$\tilde{v}_n=\Big[\frac{\int\exp\big(\sum_{j=1}^n w(s_j)\cdot\partial_{z_j}\big)\exp\big(-\frac{1}{4}S(w)\big)dw}{\int\exp\big(-\frac{1}{4}S(w)\big)dw}c(z_n)\cdots c(z_1)\Big]\Big|_{\substack{z_1=q^\natural_\omega(s_1)\\ \cdots\\ z_n=q^\natural_\omega(s_n)}}. \tag{5.6}$$

To use (5.2), let us define

$$E:=\{w:\mathbb{R}\longrightarrow\mathbb{R}^\nu, w(0)=w(t)=0\},$$

$$\langle w_1,w_2\rangle:=\int_0^t w_1(s)\cdot w_2(s)ds,$$

$$A:=-\frac{d^2}{ds^2}+\omega^2.$$

Then

$$A^{-1}w(s)=\int_0^t\frac{\mathrm{sh}(\omega s\wedge s')\,\mathrm{sh}\big(\omega(t-s\vee s')\big)}{\omega\,\mathrm{sh}(\omega t)}w(s')ds'. \tag{5.7}$$

As in quantum field theory, the operator A^{-1} can be viewed as a propagator. Since

$$\sum_{j=1}^n w(s_j)\cdot\partial_{z_j}=\langle\sum_{j=1}^n\delta_{s_j}\partial_{z_j},w\rangle$$

and as

$$\big\langle A^{-1}\sum_{j=1}^n\delta_{s_j}\partial_{z_j},\sum_{j=1}^n\delta_{s_j}\partial_{z_j}\big\rangle=\sum_{j,k=1}^n\Omega^\natural_{j,k}\partial_{z_j}\cdot\partial_{z_k},$$

(5.2) gives

$$\frac{\int\exp\big(\sum_{j=1}^n w(s_j)\cdot\partial_{z_j}\big)\exp\big(-\frac{1}{4}S(w)\big)dw}{\int\exp\big(-\frac{1}{4}S(w)\big)dw}=\exp\Big(\sum_{j,k=1}^n\Omega^\natural_{j,k}\partial_{z_j}\cdot\partial_{z_k}\Big). \tag{5.8}$$

The deformation formula (5.1) is then a consequence of (5.6) and (5.8).

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